# SIPTA Summer School 2016

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29 August – 2 September 2016

Monday 9:00-12:30

# Part 1
# Introduction

by Matthias C. M. Troffaes

# Outline: Introduction

- Welcome! (9am)
- Brainstorm: what is uncertainty, what is information (9:20am)
- Introductory applications (9:40am)
- Breakout discussion about how we deal with uncertainty (9:50am)
- Break (10:30am)
- Foundations of imprecise probability (11am)
  - Requirements
  - Uncertainty via Probability
  - Dealing With Severe Uncertainty
  - Formal Definitions
  - Sensitivity Interpretation
  - Behavioural Interpretation
  - Summary and Outlook
- Exercises (11:45am)
- Lunch (12:30pm)

# Welcome!

- lovely to meet you all!
- badges
- lecturers
- details on coffee breaks and lunches
- your presentations for the afternoon
- what to do in case of fire

# Plan for this week

- Fluffy Monday: Matthias
- Robust Tuesday: Gero & Edoardo & Ullrika
- Theoretical Wednesday: Matthias & Gero
- Applied Thursday: Ullrika & Edoardo + feedback + gala
- Reflective Friday: Edoardo + reflection + brewery

interactivity encouraged! ask questions any time!!
50% exercises: we will have a lot of fun!!

(we really want to make you feel like you deserved that brewery trip)

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# Brainstorm

- 4 or 5 groups of 4 or 5 people
- try to answer each of the questions very briefly (10 minutes)
- each group to present answers (2 minutes per group)

## Questions

1. what is uncertainty? what types of uncertainty are there?
2. when does uncertainty occur?
3. when should you ignore uncertainty, and when should you not?
4. why is uncertainty sometimes a problem?
5. how can you quantify uncertainty? what are the methods?

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# Weather

- consider the weather $x$ in Durham $y$ days from now
- assume you are offered today the following gamble, where $\alpha$ is a real-valued parameter

| outcome of $x$ | payoff (in €) |
|---|---|
| rain | $2-\alpha$ |
| clouds but dry | $-\alpha$ |
| sun | $-2-\alpha$ |

- consider the gamble in each case $\alpha=-2$, $\alpha=0$, and $\alpha=2$, for $y=1$, $y=3$, and $y=7$
- which gambles would you accept? for what other values of $\alpha$ might you accept the gamble?

# Reliability network

## Definition

network =

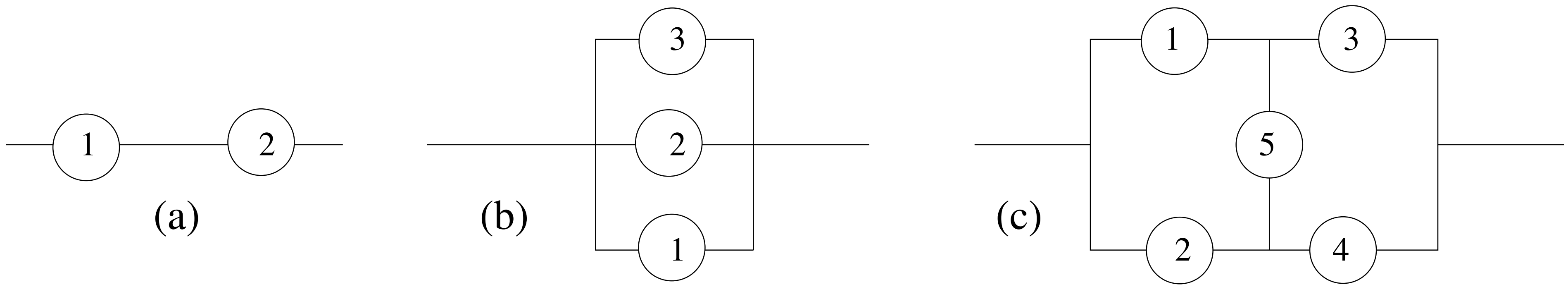
set of nodes

and arcs between some pairs of the nodes

## Definition

reliability network =

- nodes are components, each is either working or not
- entire system works when you can from left to right
  through working components only




what can you say about the reliability of the system?

# Breast cancer

- breast tissue biopsy expensive and painful, want to avoid
- alternative possible measurements
  - BIRADS assessment (from expert)
  - age (from patient)
  - shape (from X-ray)
  - margin (from X-ray)
  - density (from X-ray)
- can we rely on screening and expert information? cost-effectiveness?

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# Breakout discussion

- 4 or 5 groups of 4 or 5 people
- each group does the tasks (20 minutes)
- each group to present results (5 minutes per group)
- discussion (10 minutes)

## Tasks

(A) pick one (or more) of the examples (weather/network/cancer)
(B) identify relevant model variables
(C) how would you quantify your uncertainty about each variable?
(D) do you expect issues when quantifying these uncertainties?
(E) how you might deal with these issues?

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# Requirements for an Uncertainty Model

## Operational

How can uncertainty be reliably

- measured?
- communicated?

## Inference

How can we use our theory of uncertainty for

- statistical reasoning?
- decision making?

in the following: 'baby version' of the theory of coherent lower previsions
for the full version, see Miranda [14] or Walley [26]

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# Events: Definition

## Definition

An event is a statement that may, or may not, hold
—typically, something that may happen in the future.

Notation: $A$, $B$, $C$, ...

## Examples

- tomorrow, it will rain
- in the next year, at most 3 components will fail

how to express our uncertainty regarding events?

# Probability: Definition

## Definition

The probability of an event is a number between 0 and 1.

Notation: $P(A)$, $P(B)$, $P(C)$, ...

## Examples

- for $A =$ 'tomorrow, it will rain'
  my probability $P(A)$ is 0.2
- for $B =$ 'in the next year, at most 3 components will fail'
  my probability $P(B)$ is 0.0173

what does this number actually mean?
how would you measure it?

# Probability: Interpretation

## Interpretation: Trivial Cases

$P(A) = 0 \iff A$ is practically impossible logically?

$P(A) = 1 \iff A$ is practically certain

what about values between 0 and 1, such as $P(A) = 0.2$?

## Interpretation: General Case

- it's a frequency
- it's a betting rate
- it's something else

# Probability: Frequency Interpretation

$P(A) = 0.2$ means:

- in 1 out of 5 times, it rains tomorrow
  **nonsense, because tomorrow is not repeatable!**
- on a 'day like this', in 1 out of 5 times, it rains the next day

## Frequency Interpretation

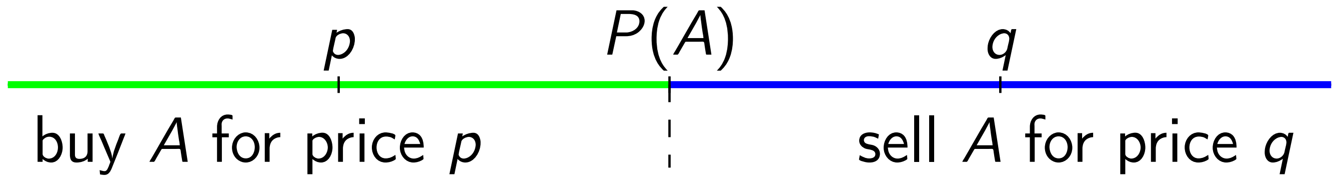
- needs reference class, only for repeatable events
- needs plenty of data

! aleatory

# Probability: Betting Interpretation

$P(A) = 0.2$ means:

- I would *now* pay at most €0.2
  if *tomorrow* I am paid €1 in case it rains
- I would *tomorrow* pay €1 in case it rains
  if I am *now* paid at least €0.2




## Betting Interpretation

+ no reference class, works also for one-shot events
- needs plenty of elicitation or plenty of data

! epistemic

# Outline: Introduction

# Dealing With Severe Uncertainty

in case of partial elicitation and/or sparse data
it may be hard to specify an exact probability
**but you may still confidently bound your probability**

this becomes more and more relevant
as problems become larger and larger

# Dealing With Severe Uncertainty: Bounding Methods

## Confidence intervals

- choice of confidence level $\alpha$?
- prosecutor's fallacy

+ no prior needed, only likelihood

## Credible intervals

- choice of credible level $\alpha$?
- choice of prior?
- dealing with prior ignorance?

+ no prosecutor's fallacy

## Interval probability (bounding probabilities directly)

- choice of prior bounds?

+ no confidence/credible level issues
+ no prior ignorance issues
+ no prosecutor's fallacy

# Detour: Prosecutor's Fallacy

Here is a (slightly modified) extract from the UK Crown Prosecution Service recommendations on interpretation of statistical evidence:

> *The fallacy is to equate the rarity of the DNA profile to the likelihood of guilt. Expressing the statistical conclusion in the wrong terms may mislead the jury.*
> *For example, the scientist's evidence states: "The chances of finding the matching profiles if this blood stain had originated from a man in the general population other than and unrelated to the defendant is 1 in 5 million."*
> *The prosecutor or judge translates this into any of the following statements;*
>
> - *the likelihood that the defendant is guilty is 5 million to 1 or;*
> - *the blood stain is 5 million times more likely to have come from the defendant than any other man;*
> - *it is 5 million to 1 against that a man other than the defendant left the blood stain.*
>
> *All the statements in the above paragraph are misleading and require evidence other than the scientist's finding to support them.*

Why are those statements misleading?

# Detour: Prosecutor's Fallacy

## Probabilistic Analysis

- $E$ = event that the DNA evidence is left
- $G$ = event that suspect with DNA match is guilty
- DNA evidence = what is the chance that the DNA at the scene of the crime was left by someone taken at random from the UK population $= P(E|G^c)$
- fallacy = confusion of $P(E|G^c)$ with $P(G|E)$!!

  confidence intervals:
  use $P(\text{data} \mid \text{parameter})$ for inference about parameter—instead of $P(\text{parameter} \mid \text{data})$

  not the same, but related via Bayes theorem:

$$P(G|E) = \frac{P(E|G)P(G)}{P(E|G)P(G) + P(E|G^c)P(G^c)}.$$

- for simplicity, assume $P(E|G) = 1$
- we still need to know $P(G)$ = prior probability of suspect's guilt

# Detour: Prosecutor's Fallacy

## Example

- burglary near Buckingham palace
- two suspects with matching DNA
  - queen
  - master criminal seen in vicinity
- suppose $P(E|G^c) = 1/1m$ (same for queen & criminal!)
- queen has $P(G) = 1/50m$,
- master criminal has $P(G) = 1/100$

$$\text{queen: } P(G|E) = \frac{1/50m}{1/50m + 1/1m \times (1 - 1/50m)} = 0.0196$$

$$\text{criminal: } P(G|E) = \frac{1/100}{1/100 + 1/1m \times 99/100} = 0.99990$$

# Lower and Upper Probability: Definition

### Definition

The lower and upper probability of an event are numbers between 0 and 1.

Notation: $\underline{P}(A)$, $\overline{P}(A)$, ...

### Examples

- for $A =$ 'tomorrow, it will rain'
  my lower probability $\underline{P}(A)$ is 0.1
  my upper probability $\overline{P}(A)$ is 0.4

what do these numbers actually mean?
how would you measure it?

# Lower and Upper Probability: Betting Interpretation

$\underline{P}(A) = 0.1$ and $\overline{P}(A) = 0.4$ means:

- I would *now* pay at most €0.1
  if *tomorrow* I am paid €1 in case it rains
- I would *tomorrow* pay €1 in case it rains
  if I am *now* paid at least €0.4


## Betting Interpretation

\+ no reference class, works also for one-shot events

\+ works with partial elicitation and/or sparse data

! epistemic

frequency interpretation?

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# Events: Formal Definition

### Definition

The possibility space $\Omega$ is
the set of all possible outcomes of the problem at hand.

### Example

interested in reliability of a system with 5 components
e.g. number of components that fail in the next year
$\Omega = \{0, 1, 2, 3, 4, 5\}$

### Definition

An event is a subset of $\Omega$. Notation: $A$, $B$, $C$, ...

### Example

in the next year, at most 3 components will fail
would be represented by the event $A = \{0, 1, 2, 3\}$

# Lower and Upper Probabilities: Formal Definition

## Definition

A lower probability $\underline{P}$
maps *every* event $A \subseteq \Omega$ to a real number $\underline{P}(A)$.

The upper probability $\overline{P}$ is simply defined as
$\overline{P}(A) = 1 - \underline{P}(A^c)$, for all $A \subseteq \Omega$

- $A^c$ = complement (or negation) of $A$ = all elements *not* in $A$

## Example

complement of ‘at most 3 components will fail’ ($A = \{0, 1, 2, 3\}$)
is ‘at least 4 components will fail’ ($A^c = \{4, 5\}$)

- the identity $\overline{P}(A) = 1 - \underline{P}(A^c)$ is implied by the betting interpretation

see exercises

- *every* event $\leftrightarrow$ sparse data?
  we can always set $\underline{P}(A) = 0$ and $\underline{P}(A^c) = 0$ ( $\iff$ $\overline{P}(A) = 1$)!

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# Lower and Upper Probabilities: Credal Set

**Definition**

A probability measure $P$
maps *every* event $A \subseteq \Omega$ to a number $P(A)$ in $[0, 1]$
and satisfies

- $P(\emptyset) = 0$,
- $P(\Omega) = 1$, and
- $P(A) = \sum_{\omega \in A} P(\{\omega\})$.

**Definition**

The credal set $\mathcal{M}$ of $\underline{P}$
is the set of all probability measures $P$ for which
$\underline{P}(A) \leq P(A) \leq \overline{P}(A)$ for all $A \subseteq \Omega$.

# Lower and Upper Probabilities: Avoiding Sure Loss, Natural Extension, and Coherence

## Definition

We say that $\underline{P}$ avoid sure loss if its credal set $\mathcal{M}$ is non-empty.

## Definition

If $\underline{P}$ avoids sure loss, its natural extension $\underline{E}$ is defined, for all $A \subseteq \Omega$, as:

$$\underline{E}(A) = \min_{P \in \mathcal{M}} P(A) \qquad \left(\text{or equivalently, } \overline{E}(A) = \max_{P \in \mathcal{M}} P(A)\right)$$

## Definition

We say that $\underline{P}$ is coherent if it avoids sure loss, and, for all $A \subseteq \Omega$:

$$\underline{P}(A) = \underline{E}(A) \qquad \left(\text{or equivalently, } \overline{P}(A) = \overline{E}(A)\right)$$

# Lower and Upper Probabilities: Sensitivity Interpretation

## Sensitivity Interpretation of $\underline{P}$

One of the probability measures $P$ in the credal set $\mathcal{M}$ is the correct one,
*but we do not know which*.

- $\underline{P}$ is coherent precisely when it is uniquely determined by $\mathcal{M}$
- if $\underline{P}$ is not coherent, but avoids sure loss then its natural extension $\underline{E}$ corrects $\underline{P}$

**crucial: no distribution over $\mathcal{M}$ assumed!**
(why not?)

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# Desirability: Example

### Example

remember, for $A =$ 'tomorrow, it will rain', $\underline{P}(A) = 0.1$ means:

- I would *now* pay at most €0.1
  if *tomorrow* I am paid €1 in case it rains

$\Longleftrightarrow$

- I would *now* accept the payoff $f =$

| $\omega$ | rain | no rain |
| --- | --- | --- |
| $f(\omega)$ | €0.9 | −€0.1 |

  to be paid out to me *tomorrow*

- $f$ is said to be a desirable gamble
- notation: $f = I_A - 0.1$
- $I_A$ is the indicator of $A$:

| $\omega$ | rain | no rain |
| --- | --- | --- |
| $I_A(\omega)$ | 1 | 0 |

# Desirability: Definition

## Definition

A gamble is a real-valued function on $\Omega$.

## Definition

A gamble is desirable to you
if you accept it now as a payoff to be paid out when $\omega$ is revealed.

## Observation

- specifying $\underline{P}(A)$ is equivalent to declaring $I_A - \underline{P}(A)$ to be desirable
- specifying $\overline{P}(A)$ is equivalent to declaring $\overline{P}(A) - I_A$ to be desirable

# Desirability Axioms: Rationality

**Accept Sure Gain**

If $f(\omega) \geq 0$ for all $\omega$ then $f$ is desirable.

**Avoid Sure Loss**

If $f(\omega) < 0$ for all $\omega$ then $f$ is not desirable.

**Accept Scaled Bets**

If $f$ is desirable, and $\lambda$ is a strictly positive real number, then $\lambda f$ is desirable.

**Accept Combined Bets**

If $f$ and $g$ are desirable then $f + g$ is desirable.

- These four axioms are called the axioms of desirability.
- **Everything follows from just these four axioms!**

# Lower and Upper Probabilities: Behavioural Interpretation of Avoiding Sure Loss

## Example

consider again $\Omega = \{0, 1, 2, 3, 4, 5\}$ (number of components failing)

- suppose I specify $\underline{P}(\{0, 1, 2\}) = 0.45$ and $\underline{P}(\{3, 4, 5\}) = 0.6$
- equivalently, I declare $I_{\{0,1,2\}} - 0.45$ and $I_{\{3,4,5\}} - 0.6$ to be desirable
- by [Accept Combined Bets] $I_{\{0,1,2\}} - 0.45 + I_{\{3,4,5\}} - 0.6$ is desirable
- but $I_{\{0,1,2\}} - 0.45 + I_{\{3,4,5\}} - 0.6 = 1 - 0.45 - 0.6 = -0.05$ whence I violate [Avoid Sure Loss]

## Observation

In the above example, the credal set $\mathcal{M}$ of $\underline{P}$ is actually empty.
Is this a coincidence?

# Lower and Upper Probabilities: Behavioural Interpretation of Avoiding Sure Loss

### Definition

$\underline{P}$ is said to avoid sure loss whenever, for all $\lambda_B \geq 0$,

$$\sup \left[ \sum_{B \subseteq \Omega} \lambda_B (I_B - \underline{P}(B)) \right] \geq 0. \tag{1}$$

### Theorem

*This definition is equivalent to our earlier one in terms of the credal set $\mathcal{M}$ of $\underline{P}$.*

# Lower and Upper Probabilities: Behavioural Interpretation of Natural Extension

## Example

consider again $\Omega = \{0, 1, 2, 3, 4, 5\}$ (number of components failing)

- suppose $\underline{P}(\{0\}) = 0.1$, $\underline{P}(\{1, 2\}) = 0.3$, and $\underline{P}(\{0, 1, 2\}) = 0.35$
- in particular, I declare $I_{\{0\}} - 0.1$ and $I_{\{1,2\}} - 0.3$ to be desirable
- by [Accept Combined Bets] $I_{\{0\}} - 0.1 + I_{\{1,2\}} - 0.3$ is desirable
- but $I_{\{0\}} - 0.1 + I_{\{1,2\}} - 0.3 = I_{\{0,1,2\}} - 0.4$
  whence, I am willing to pay 0.4 for $I_{\{0,1,2\}}$
  whence, my lower probability for $\{0, 1, 2\}$ should be at least 0.4
  initial assessment $\underline{P}(\{0, 1, 2\}) = 0.35$ is too conservative

## Observation

In the above example, $\underline{E}(\{0, 1, 2\}) = \min_{P \in \mathcal{M}} P(\{0, 1, 2\}) = 0.4$.
Is this a coincidence?

# Lower and Upper Probabilities: Behavioural Interpretation of Natural Extension

## Definition

The natural extension $\underline{E}$ of $\underline{P}$ is defined as:

$$\underline{E}(A) = \sup \left\{ \alpha \in \mathbb{R} \colon I_A - \alpha \geq \sum_{B \subseteq \Omega} \lambda_B (I_B - \underline{P}(B)) \right\} \qquad (2)$$

## Theorem

*This definition is again equivalent to our earlier one in terms of the credal set $\mathcal{M}$ of $\underline{P}$.*

## Proof.

It is the dual linear program. □

see exercises

# Lower and Upper Probabilities: Behavioural Interpretation of Coherence

## Definition

$\underline{P}$ is said to coherent whenever it avoids sure loss, and for all $A \subseteq \Omega$ and all $\lambda_B \geq 0$,

$$\sup \left[ \sum_{B \subseteq \Omega} \lambda_B \Big( I_B - \underline{P}(B) \Big) - \Big( I_A - \underline{P}(A) \Big) \right] \geq 0. \tag{3}$$

interpretation: if the above inequality is violated,
then one can correct $\underline{P}(A)$ by means of natural extension
specifically, one can construct a price for $I_A$ higher than $\underline{P}(A)$
via the desirable gamble $\sum_{B \subseteq \Omega} \lambda_B (I_B - \underline{P}(B))$

## Theorem

*This definition is equivalent to our earlier one in terms of the credal set $\mathcal{M}$ of $\underline{P}$.*

# Dogma of precision

(term coined by Peter Walley)

## Can we recover standard probability theory?

Can we add something to the desirability axioms to get standard probability theory with unique distributions for every variable?

**Yes, the following axiom does the trick!**

### Fair Price

For every gamble $f$, there is a unique number $E(f)$ such that
$f - \alpha$ is desirable for all $\alpha < E(f)$ and
$\alpha - f$ is desirable for all $\alpha > E(f)$

- $E(f) =$ 'fair price' or 'expectation'
- we argue that the 'fair price' assumption is often too strong: if you have very little information about $f$, it may be very hard to identify the number $E(f)$ which satisfies the conditions of the axiom!

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# Summary and Outlook

- How to do statistics with partial elicitation and sparse data?
- Use of lower and upper probability appears, at least naively, to be a simple way of dealing with severe uncertainty.
- Sensitivity interpretation via credal set.
  Behavioural interpretation via desirability.
- How do you actually get the lower and upper bounds 'just from' data?

Where to go from here? e.g. Miranda's survey paper on lower previsions [14]

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# Exercise 1: Betting Interpretation

Recall that, for $A$ = 'tomorrow, it will rain', $\overline{P}(A) = 0.4$ means:

- I would *tomorrow* pay €1 in case it rains
  if I am *now* paid at least €0.4

Show that this transaction is identical to the following:

- I would *now* pay at most €0.6
  if *tomorrow* I am paid €1 in case it does not rain

(In other words, this means that $\underline{P}(A^c) = 0.6$, or in other words, we have established the identity $\overline{P}(A) = 1 - \underline{P}(A^c)$.)

# Exercise 2: Avoiding Sure Loss

Consider the possibility space

$$\Omega = \{a, b\}$$

where $a$ corresponds to tomorrow there being rain, and $b$ to the opposite.

*Which of the following four lower probabilities avoid sure loss?*

1.

| $A$ | $\emptyset$ | $\{a\}$ | $\{b\}$ | $\Omega$ |
|---|---|---|---|---|
| $\underline{P}(A)$ | 0 | 0 | 0 | 1 |

2.

| $A$ | $\emptyset$ | $\{a\}$ | $\{b\}$ | $\Omega$ |
|---|---|---|---|---|
| $\underline{P}(A)$ | $-1$ | 0 | 0 | 0 |

3.

| $A$ | $\emptyset$ | $\{a\}$ | $\{b\}$ | $\Omega$ |
|---|---|---|---|---|
| $\underline{P}(A)$ | 0 | 0.2 | 0.3 | 1 |

4.

| $A$ | $\emptyset$ | $\{a\}$ | $\{b\}$ | $\Omega$ |
|---|---|---|---|---|
| $\underline{P}(A)$ | 0 | 0.8 | 0.7 | 1 |

# Exercise 3: Credal Set and Natural Extension

*Calculate the credal set and, permitting, the natural extension, of the lower probabilities given in Exercise 2.*

# Exercise 4: Coherence

*Which of the lower probabilities of Exercise 2 are coherent?*

# (*) Exercise 5: Avoiding Sure Loss

Suppose that $\underline{P}(A) > \overline{P}(A)$ for some event $A \subseteq \Omega$.

*Show that* $\underline{P}$ *does not avoid sure loss.*

# (*) Exercise 6: Coherence

Suppose that $\underline{P}(A) > \underline{P}(B)$ for some events $A \subsetneq B \subseteq \Omega$.

*Show that $\underline{P}$ is not coherent.*

# (**) Exercise 7: Avoiding Sure Loss

Consider the possibility space

$$\Omega = \{a, b\}$$

where $a$ corresponds to tomorrow there being rain, and $b$ to the opposite.
*For this case, prove that a lower probability $\underline{P}$ avoid sure loss if and only if all of the following conditions are satisfied:*

- $\underline{P}(\emptyset) \leq 0$
- $\underline{P}(\Omega) \leq 1$
- $\underline{P}(\{a\}) + \underline{P}(\{b\}) \leq 1$

# (**) Exercise 8: Coherence

Consider the possibility space

$$\Omega = \{a, b\}$$

where $a$ corresponds to tomorrow there being rain, and $b$ to the opposite.
*For this case, prove that a lower probability $\underline{P}$ is coherent if and only if all of the following conditions are satisfied:*

- $\underline{P}(\emptyset) = 0$
- $\underline{P}(\Omega) = 1$
- $\underline{P}(\{a\}) \geq 0$
- $\underline{P}(\{b\}) \geq 0$
- $\underline{P}(\{a\}) + \underline{P}(\{b\}) \leq 1$

# (***) Exercise 9: Natural Extension

Natural extension, in terms of the credal set of $\underline{P}$, can be written as a linear program:

- minimize $\sum_{\omega \in A} p(\omega)$
- subject to

$$\begin{aligned} p(\omega) &\geq 0 && \text{for all } \omega \in \Omega \\ \sum_{\omega \in \Omega} p(\omega) &= 1 \\ \sum_{\omega \in B} p(B) &\geq \underline{P}(B) && \text{for all } B \subseteq \Omega \end{aligned}$$

*Show that the expression for natural extension in terms of desirability corresponds to the dual linear program of the expression of natural extension in terms of the credal set.*

## Dual Linear Program

'minimize $b^T x$ subject to $Ax \geq c$, $x \geq 0$'
is equivalent to
'maximize $c^T y$ subject to $A^T y \leq b$, $y \geq 0$'

# Outline: Introduction

- Welcome! (9am)
- Brainstorm: what is uncertainty, what is information (9:20am)
- Introductory applications (9:40am)
- Breakout discussion about how we deal with uncertainty (9:50am)
- Break (10:30am)
- Foundations of imprecise probability (11am)
  - Requirements
  - Uncertainty via Probability
  - Dealing With Severe Uncertainty
  - Formal Definitions
  - Sensitivity Interpretation
  - Behavioural Interpretation
  - Summary and Outlook
- Exercises (11:45am)
- Lunch (12:30pm)

Monday 14:00-17:30

# Part 2
# Student presentations

by you

# Outline: Student presentations

Student Presentations I (2pm)

Break (3:30pm)

Student Presentations II (4pm)

# Student Presentations

- Dominic
- Zahida
- Florentin
- Louis
- Mauro
- Laura
- Alexander
- Lanting
- Jesca
- Evelyn

# Outline: Student presentations

Student Presentations I (2pm)

Break (3:30pm)

Student Presentations II (4pm)

# Outline: Student presentations

Student Presentations I (2pm)

Break (3:30pm)

Student Presentations II (4pm)

# Student Presentations

- Domenico
- Chen
- Ting
- Joseph
- Jonathan
- Roberto
- Angela
- Naeima
- Manal
- Hana