Tuesday 9:00-12:30

Part 3
Robust Bayesian statistics & applications in reliability networks

by Gero Walter
Robust Bayesian statistics & applications in reliability networks

Outline

Robust Bayesian Analysis (9am)
  Why
  The Imprecise Dirichlet Model
  General Framework for Canonical Exponential Families

Exercises I (9:30am)

System Reliability Application (10am)

Break (10:30am)

Exercises II (11am)
Robust Bayesian statistics & applications in reliability networks

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Robust Bayesian Analysis: Why

- choice of prior can severely affect inferences even if your prior is ‘non-informative’
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- close relations to robust Bayes literature, e.g. [7, 19, 20]
- concerns uncertainty in the prior (uncertainty in data generating process: imprecise sampling models)
- here: focus on imprecise Dirichlet model
- if your prior is informative then prior-data conflict can be an issue [31, 29] (we’ll come back to this in the system reliability application)
Robust Bayesian Analysis: Principle of Indifference

How to construct a prior if we do not have a lot of information?

Laplace: Principle of Indifference

Use the uniform distribution.

Obvious issue: this depends on the parametrisation!

Example

An object of 1kg has uncertain volume $V$ between 1 $\ell$ and 2 $\ell$.

▶ Uniform distribution over volume $V = \Rightarrow E(V) = 1.5 \ell$.

▶ Uniform distribution over density $\rho = \Rightarrow E(1/\rho) = \int_0^{1.5} \frac{2}{\rho} d\rho = 2 (\ln 1 - \ln 0.5) = 1.39 \ell$.

The uniform distribution does not really model prior ignorance. (Jeffreys prior is transformation-invariant, but depends on the sample space and can break decision making!)
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**Example**

An object of 1kg has uncertain volume $V$ between $1\ell$ and $2\ell$.

- Uniform distribution over volume $V$ $\implies E(V) = 1.5\ell$.
- Uniform distribution over density $\rho = 1/V$ $\implies E(V) = E(1/\rho) = \int_{0.5}^{1} 2/\rho d\rho = 2(\ln 1 - \ln 0.5) = 1.39\ell$
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Robust Bayesian Analysis: Prior Ignorance via Sets of Probabilities

How to construct prior if we do not have a lot of information?

**Boole: Probability Bounding**

Use the set of all probability distributions (**vacuous model**).

Results no longer depend on parametrisation!
Robust Bayesian Analysis: Prior Ignorance via Sets of Probabilities

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Boole: Probability Bounding

Use the set of all probability distributions (vacuous model).

Results no longer depend on parametrisation!

Example

An object of 1kg has uncertain volume $V$ between $1\ell$ and $2\ell$.

- Set of all distributions over volume $V \implies E(V) \in [1, 2]$.
- Set of all distribution over density $\rho = 1/V \implies E(V) = E(1/\rho) \in [1, 2]$
Robust Bayesian Analysis: Prior Ignorance via Sets of Probabilities

Theorem

\textit{The set of posterior distributions resulting from a vacuous set of prior distributions is again vacuous, regardless of the likelihood.}

We can never learn anything when starting from a vacuous set of priors.
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We can never learn anything when starting from a vacuous set of priors.

Solution: Near-Vacuous Sets of Priors

Only insist that the prior predictive, or other classes of inferences, are vacuous.

This can be done using sets of conjugate priors [4, 5].
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The Imprecise Dirichlet Model: Definition

- introduced by Peter Walley [27, 28]
- for multinomial sampling, $k$ categories 1, 2, ..., $k$
- Bayesian conjugate analysis
  - multinomial likelihood (sample $n = (n_1, \ldots, n_k)$, $\sum n_i = n$)
    \[
    f(n \mid \theta) = \frac{n!}{n_1! \cdots n_k!} \prod_{i=1}^{k} \theta_i^{n_i}
    \]
- conjugate Dirichlet prior
  - with mean $t = (t_1, \ldots, t_k) =$ prior expected proportions
  - and parameter $s > 0$
    \[
    f(\theta) = \frac{\Gamma(s)}{\prod_{i=1}^{k} \Gamma(st_i)} \prod_{i=1}^{k} \theta_i^{st_i-1}
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\]

Definition (Imprecise Dirichlet Model)

Use the set \( \mathcal{M}^{(0)} \) of all Dirichlet priors, for a fixed \( s > 0 \), and take the infimum/supremum over \( t \) of the posterior to get lower/upper predictive probabilities/expectations.
The Imprecise Dirichlet Model: Properties

- **conjugacy:** \( f(\theta \mid n) \) again Dirichlet with parameters
  \[
  t_i^* = \frac{st_i + ni}{s+n} = \frac{s}{s+n} t_i + \frac{n}{s+n} \frac{ni}{n},
  \]
  \[
  s^* = s + n
  \]

- \( t_i^* = E(\theta_i \mid n) = P(i \mid n) \) is a weighted average of \( t_i \) and \( ni/n \), with weights proportional to \( s \) and \( n \), respectively

- \( s \) can be interpreted as a prior strength or pseudocount

- lower and upper expectations / probabilities by min and max over \( t \in \Delta \) (unit simplex)
The Imprecise Dirichlet Model: Properties

Posterior predictive probabilities

- for observing a particular category
  \[
  \underline{P}(i \mid n) = \frac{n_i}{s + n}, \quad \overline{P}(i \mid n) = \frac{s + n_i}{s + n}
  \]

- for observing a non-trivial event \( A \subseteq \{1, \ldots, k\} \)
  \[
  \underline{P}(A \mid n) = \frac{n_A}{s + n}, \quad \overline{P}(A \mid n) = \frac{s + n_A}{s + n},
  \]
  with \( n_A = \sum_{i \in A} n_i \)

Satisfies prior near ignorance:
vacuous for prior predictive \( \underline{P}(A) = 0, \overline{P}(A) = 1 \)
Inferences are independent of categorisation
(‘Representation Invariance Principle’).
The Imprecise Dirichlet Model: Why A Set of Priors?

- single prior $\implies$ dependence on categorisation
- for example, single Dirichlet prior (with $t_A = \sum_{i \in A} t_i$, $s = 2$)

$$P(A | n) = \frac{2t_A + n_A}{n + 2}$$

one red marble observed

- two categories red (R) and other (O):
  prior ignorance $\implies t_R = t_O = \frac{1}{2} \implies P(R | n) = \frac{2}{3}$
- three categories red (R), green (G), and blue (B):
  prior ignorance $\implies t_R = t_G = t_B = \frac{1}{3} \implies P(R | n) = \frac{5}{9}$
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  - prior ignorance $\implies t_R = t_G = t_B = \frac{1}{3} \implies P(R \mid n) = \frac{5}{9}$

prior ignorance + representation invariance principle
$\implies$ must use set of priors
The Imprecise Dirichlet Model: The $s$ Parameter

- $s$ can be interpreted as a **prior strength** or pseudocount
- $s$ determines **learning speed**:

$$
\overline{P}(A | n) - \underline{P}(A | n) = \frac{s}{s + n}
$$

- **no objective** way of choosing $s$, but $s = 2$ covers most Bayesian and frequentist inferences
- for $s = n$ posterior imprecision is half the prior imprecision
- for informative $t_i$ bounds, using a range of $s$ values allows the set of posteriors to reflect *prior-data conflict* (see system reliability application)
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General Framework for Canonical Exponential Families

Conjugate priors like the Dirichlet can be constructed for sample distributions (likelihood) from:

Definition (Canonical exponential family)

\[
f(x | \psi) = h(x) \exp \left\{ \psi^T \tau(x) - b(\psi) \right\}
\]

- includes multinomial, normal, Poisson, exponential, …
- \(\psi\) generally a transformation of original parameter \(\theta\)
Conjugate priors like the Dirichlet can be constructed for sample distributions (likelihood) from:

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- includes multinomial, normal, Poisson, exponential, ... 
- \( \psi \) generally a transformation of original parameter \( \theta \)

**Definition (Family of conjugate priors)**

A family of priors for i.i.d. sampling from the can. exp. family:

\[
f(\psi \mid n^{(0)}, y^{(0)}) \propto \exp \left\{ n^{(0)} \left[ \psi^T y^{(0)} - b(\psi) \right] \right\}
\]

with hyper-parameters \( n^{(0)} (\leftrightarrow s) \) and \( y^{(0)} (\leftrightarrow t \text{ from IDM}) \).
Theorem (Conjugacy)

Posterior is of the same form:

\[ f(\psi \mid n^{(0)}, y^{(0)}, x) \propto \exp \left\{ n^{(n)} [\psi^T y^{(n)} - b(\psi)] \right\} \]

where

\[ x = (x_1, \ldots, x_n) \]

\[ (s^* \leftrightarrow) \quad n^{(n)} = n^{(0)} + n \]

\[ (t^* \leftrightarrow) \quad y^{(n)} = \frac{n^{(0)}}{n^{(0)} + n} \cdot y^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{\tau(x)}{n} \]

\[ (n_i \leftrightarrow) \quad \tau(x) = \sum_{i=1}^{n} \tau(x_i) \]
General Framework for Canonical Exponential Families

- \( y^{(0)} \leftrightarrow t \) = prior expectation of \( \tau(x)/n \)
- \( n^{(0)} \leftrightarrow s \) determines spread and learning speed

usefulness of this framework for IP / robust Bayes discovered by Quaghebeur & de Cooman [18]
near-noninformative sets of priors developed by Benavoli & Zaffalon [4, 5]
for informative sets of priors, Walter & Augustin [29, 31] suggest to use parameter sets
\( \Pi^{(0)} = [n^{(0)}, n^{(0)}] \times [y^{(0)}, y^{(0)}] \)
• $y^{(0)} (\leftrightarrow t) = \text{prior expectation of } \tau(x)/n$
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General Framework: Why vary $n^{(0)}(\leftrightarrow s)$?

What if prior assumptions and data tell different stories?

Prior-Data Conflict

➤ *informative prior beliefs and trusted data* (sampling model correct, no outliers, etc.) are in conflict
➤ “the prior [places] its mass primarily on distributions in the sampling model for which the observed data is surprising” [10]
➤ there are not enough data to overrule the prior

Example: IDM with $k = 2 \implies$ Imprecise Beta Model
Imprecise Beta Model (IBM)

- binomial likelihood (observing $x$ successes in $n$ trials)

$$f(x | \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}$$

- conjugate Beta prior
  - with mean $y^{(0)} = \text{prior expected probability of success}$
  - and prior strength parameter $n^{(0)} > 0$

$$f(\theta) \propto \theta^{n^{(0)} y^{(0)} - 1} (1 - \theta)^{n^{(0)} (1 - y^{(0)}) - 1}$$
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- informative set of priors: Use the set $\mathcal{M}^{(0)}$ of Beta priors with $y^{(0)} \in [\underline{y}^{(0)}, \overline{y}^{(0)}]$ and
  - $n^{(0)} > 0$ fixed, or
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- \(E(\theta \mid x) = y^{(n)}\) is a weighted average of \(E(\theta) = y^{(0)}\) and \(\frac{x}{n}!\)

- \(\text{Var}(\theta \mid x) = \frac{y^{(n)} (1 - y^{(n)})}{n^{(n)} + 1}\) decreases with \(n!\)
Imprecise Beta Model with \( n^{(0)} \) fixed

no conflict:

prior \( n^{(0)} = 8, \ y^{(0)} \in [0.7, 0.8] \)
data \( s/n = 12/16 = 0.75 \)
Imprecise Beta Model with $n^{(0)}$ fixed

**no conflict:**

prior $n^{(0)} = 8$, $y^{(0)} \in [0.7, 0.8]$

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$\uparrow$

prior data conflict:

prior $n^{(0)} = 8$, $y^{(0)} \in [0.2, 0.3]$

data $s/n = 16/16 = 1$

$\downarrow$

$n^{(n)} = 24$, $y^{(n)} \in [0.73, 0.77]$
Imprecise Beta Model with $n^{(0)}$ fixed

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Imprecise Beta Model with $n^{(0)}$ interval

no conflict:
prior $n^{(0)} \in [4, 8]$, $y^{(0)} \in [0.7, 0.8]$
data $s/n = 16/16 = 1$

$y(n) \in [0.73, 0.86]$
“banana” shape
Imprecise Beta Model with $n^{(0)}$ interval

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“spotlight” shape
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\[ y(n) \in [0.73, 0.77] \]
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prior-data conflict:
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Imprecise Beta Model with $n^{(0)}$ interval

No conflict:
Prior $n^{(0)} \in [4, 8]$, $y^{(0)} \in [0.7, 0.8]$
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Prior-data conflict:
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$y^{(n)} \in [0.73, 0.86]$
“banana” shape
Robust Bayesian Analysis: Other Models

- How to define sets of priors $\mathcal{M}^{(0)}$ is a crucial modeling choice
- Sets $\mathcal{M}^{(0)}$ via parameter sets $\Pi^{(0)}$ seem to work better than other models discussed in the robust Bayes literature:
  - Neighbourhood models
    - set of distributions ‘close to’ a central distribution $P_0$
    - example: $\varepsilon$-contamination class:
      \[ \{ P : P = (1 - \varepsilon)P_0 + \varepsilon Q, Q \in \mathcal{Q} \} \]
    - not necessarily closed under Bayesian updating
How to define sets of priors $\mathcal{M}^{(0)}$ is a crucial modeling choice.

Sets $\mathcal{M}^{(0)}$ via parameter sets $\Pi^{(0)}$ seem to work better than other models discussed in the robust Bayes literature:

- **Neighbourhood models**
  - set of distributions ‘close to’ a central distribution $P_0$
  - example: $\varepsilon$-contamination class:
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- **Density ratio class / interval of measures**
  - set of distributions by bounds for the density function $f(\theta)$:
    \[
    \mathcal{M}_{l,u}^{(0)} = \{ f(\theta) : \exists c \in \mathbb{R}_{>0} : l(\theta) \leq cf(\theta) \leq u(\theta) \}
    \]
    - posterior set is bounded by updated $l(\theta)$ and $u(\theta)$
    - $u(\theta)/l(\theta)$ is constant under updating
      - size of the set does not decrease with $n$
      - very vague posterior inferences
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Exercise: Update step in the IBM

Consider first a single Beta prior with parameters \( n^{(0)} \) and \( y^{(0)} \).

Discuss how \( \mathbb{E}[\theta \mid x] = y^{(n)} \) and \( \text{Var}(\theta \mid x) = \frac{y^{(n)}(1-y^{(n)})}{n^{(n)}+1} \) behave when

(i) \( n^{(0)} \to 0 \),  (ii) \( n^{(0)} \to \infty \),  (iii) \( n \to \infty \) when \( x/n = \text{const.} \).

What do the results for \( \mathbb{E}[\theta \mid x] \) and \( \text{Var}(\theta \mid x) \) imply for the shape of \( f(\theta \mid x) \)?

When considering a set of Beta priors with

\[
(n^{(0)}, y^{(0)}) \in \Pi^{(0)} = [n^{(0)}, \bar{n}^{(0)}] \times [y^{(0)}, \bar{y}^{(0)}],
\]

what does the weighted average structure of \( y^{(n)} \) tell you about the updated set of parameters? (Remember the prior-data conflict example!)
Exercise: Update step for canonically constructed priors

The canonically constructed prior for an i.i.d. sample distribution from the canonical exponential family

\[ f(x | \psi) = \left( \prod_{i=1}^{n} h(x_i) \right) \exp \left\{ \psi^T \tau(x) - nb(\psi) \right\} \]

is given by

\[ f(\psi | n(0), y(0)) \propto \exp \left\{ n(0) \left[ \psi^T y(0) - b(\psi) \right] \right\}, \]

and the corresponding posterior by

\[ f(\psi | n(0), y(0), x) \propto \exp \left\{ n(n) \left[ \psi^T y(n) - b(\psi) \right] \right\}, \]

where \[ y(n) = \frac{n(0)}{n(0) + n} \cdot y(0) + \frac{n}{n(0) + n} \cdot \frac{\tau(x)}{n} \]

and \[ n(n) = n(0) + n. \]

Confirm the expressions for \( y(n) \) and \( n(n) \).
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Reliability block diagrams:

- system consists of components $k$ (different types $k = 1, \ldots, K$)
- each $k$ either works or not
- system works when there is a path using only working components
Reliability block diagrams:

- system consists of components (different types $k = 1, \ldots, K$)
- each $k$ either works or not
- system works when there is a path using only working components

We want to learn about the system reliability $R_{sys}(t) = P(T_{sys} > t)$ (system survival function)

based on

- component test data:
  
  $n_k$ failure times for components of type $k$, $k = 1, \ldots, K$

- cautious assumptions on component reliability:
  expert information, e.g. from maintenance managers and staff
Nonparametric Component Reliability

Functioning probability $p_t^k$ of component $k$ for each time $t \in T = \{t'_1, t'_2, \ldots\}$

- discrete component reliability function

$R^k(t) = p_t^k, \ t \in T$. 

![Graph showing the functioning probability $p_t^k$ over different time intervals.](image)
Nonparametric Component Reliability

Functioning probability $p^k_t$ of $k$ for each time $t \in \mathcal{T} = \{t'_1, t'_2, \ldots\}$

- discrete component reliability function $R^k(t) = p^k_t$, $t \in \mathcal{T}$. 
Nonparametric Component Reliability

Functioning probability $p_t^k$ of type k for each time $t \in \mathcal{T} = \{t_1', t_2', \ldots\}$

- discrete component reliability function $R^k(t) = p_t^k$, $t \in \mathcal{T}$. 

![Graph showing the decay of $p_t^k$ with time $t$]
Nonparametric Component Reliability

Functioning probability $p_t^k$ of $k$ for each time $t \in T = \{t_1', t_2', \ldots\}$

- discrete component reliability function $R^k(t) = p_t^k$, $t \in T$. 

### Graph

- The graph shows the functioning probability $p_t^k$ over time $t$.
- $p_t^k$ decreases as $t$ increases, indicating a decrease in reliability over time.
Nonparametric Component Reliability

Functioning probability $p_t^k$ of for each time $t \in \mathcal{T} = \{t_1', t_2', \ldots\}$

- discrete component reliability function $R^k(t) = p_t^k$, $t \in \mathcal{T}$.

use Imprecise Beta Model to estimate $p_t^k$'s:

- Set of Beta priors for each $p_t^k$:
  $p_t^k \sim \text{Beta}(n_{k,t}^{(0)}, y_{k,t}^{(0)})$ with
  $(n_{k,t}^{(0)}, y_{k,t}^{(0)}) \in \Pi_{k,t}^{(0)} = [n_{k,t}^{(0)}, \overline{n}_{k,t}^{(0)}] \times [y_{k,t}^{(0)}, \overline{y}_{k,t}^{(0)}]$
  $\mathcal{M}_{k,t}^{(0)}$ can be near-vacuous by setting $[y_{k,t}^{(0)}, \overline{y}_{k,t}^{(0)}] = [0, 1]$
  $\mathcal{M}_{k,t}^{(n)}$ will reflect prior-data conflict for informative $\mathcal{M}_{k,t}^{(0)}$

- failure times $t^k = (t_1^k, \ldots, t_{n_k}^k)$ from component test data:
  number of type $k$ components functioning at $t$:
  $S_t^k \mid p_t^k \sim \text{Binomial}(p_t^k, n_k)$
Component Reliability with Sets of Priors

\[
\begin{align*}
[n(0), n(0)] &= [1, 2] \\
[y(0), \bar{y}(0)] &= \uparrow
\end{align*}
\]
Component Reliability with Sets of Priors

\[
\begin{bmatrix}
  n^{(0)}, \bar{n}^{(0)} \\
  y^{(0)}, \bar{y}^{(0)}
\end{bmatrix} = [1, 2] \\
[0, 1)
\]
Component Reliability with Sets of Priors

\[ [\underline{n}^{(0)}, \overline{n}^{(0)}] = [1, 8] \]
\[ [\underline{y}^{(0)}, \overline{y}^{(0)}] = \uparrow \downarrow \]
Closed form for the system reliability via the survival signature:

\[ R_{\text{sys}} \left( t \mid \bigcup_{k=1}^{K} \{ n_{k,t}^{(0)}, y_{k,t}^{(0)}, t^k \} \right) = P \left( T_{\text{sys}} > t \mid \cdots \right) \]

\[ = \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \ldots, l_K) \prod_{k=1}^{K} P(C^k_t = l_k \mid n_{k,t}^{(0)}, y_{k,t}^{(0)}, t^k) \]
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Survival signature [8] \( \Phi(l_1, \ldots, l_K) = P(\text{system functions} | \{ l_k \text{'s function} \}_{1:K}) \)

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R_{sys}(t \mid \bigcup_{k=1}^{K} \{ n_{k,t}^{(0)}, y_{k,t}^{(0)}, t^k \}) = P(T_{sys} > t \mid \cdots )
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System Reliability

Closed form for the system reliability via the survival signature:

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[Diagram of a system with nodes 1, 2, 3, and 1, 1]
Closed form for the system reliability via the survival signature:

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R_{sys}(t \mid \bigcup_{k=1}^{K} \{ n_{k,t}^{(0)}, y_{k,t}^{(0)}, t^k \}) = P(T_{sys} > t \mid \cdots )
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| \( l_1 \) | \( l_2 \) | \( l_3 \) | \( \Phi \) | | \( l_1 \) | \( l_2 \) | \( l_3 \) | \( \Phi \) |
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| 3 | 0 | 1 | 1 | 3 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 | 4 | 1 | 1 | 1 |
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- Closed form for the system reliability via the survival signature:

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\[
\Phi(0, 0, 1) = 0 \quad \Phi(1, 1, 1) = 0 \quad \Phi(2, 1, 1) = 2/3 \quad \Phi(3, 1, 1) = 1 \quad \Phi(4, 1, 1) = 1
\]
System Reliability

Closed form for the system reliability via the survival signature:

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Post. pred. probability that in a new system, \( l_k \) of the \( m_k \) \((k)\)’s function at time \( t \):

\[
\binom{m_k}{l_k} \int [P(T < t \mid p_t^k)]^{l_k} \left[ P(T \geq t \mid p_t^k) \right]^{m_k - l_k} f(p_t^k \mid n_{k,t}^{(0)}, y_{k,t}^{(0)}, t^k) \, dp_t^k
\]

analytical solution for integral:

\( C_t^k \mid n_{k,t}^{(0)}, y_{k,t}^{(0)}, \bar{t}^k \sim \text{Beta-Binom.} \)
Bounds for $R_{sys}(t \mid \bigcup_{k=1}^{K} \{ n_{k,t}^{(0)}, y_{k,t}^{(0)}, t^k \})$ over $\prod_{k,t}^{(0)}$, $k = 1, \ldots, K$:

1. $\min R_{sys}(\cdot)$ by $y_{k,t}^{(0)} = y_{k,t}^{(0)}$ for any $n_{k,t}^{(0)}$ [30, Theorem 1]

2. $\min R_{sys}(\cdot)$ for $n_{k,t}^{(0)}$ or $\bar{n}_{k,t}^{(0)}$ according to simple conditions [30, Theorem 2 & Lemma 3]

3. Numeric optimization over $[n_{k,t}^{(0)}, \bar{n}_{k,t}^{(0)}]$ in the very few cases where Theorem 2 & Lemma 3 do not apply

4. Implemented in R package ReliabilityTheory [1]
System Reliability Bounds

![Graphs showing survival probability over time for different components (M, H, C, P) with prior and posterior distributions marked.](image)
Robust Bayesian statistics & applications in reliability networks

Outline

Robust Bayesian Analysis (9am)
  Why
  The Imprecise Dirichlet Model
  General Framework for Canonical Exponential Families

Exercises I (9:30am)

System Reliability Application (10am)

Break (10:30am)

Exercises II (11am)
Robust Bayesian statistics & applications in reliability networks

Outline

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Exercises II (11am)
Exercise: Try it yourself!

- Download the development version of the package ReliabilityTheory:
  
  ```r
  library("devtools")
  install_github("louisaslett/ReliabilityTheory")
  
  (The author is present!)
  ```


- Exercise 1 considers a single component only.
- Exercise 2 considers a system with several types of components.
Exercise: Technical Hints

▶ Keeping the number of time points low (e.g., $|\mathcal{T}| = 50$) keeps computation time short.

▶ Setting lower and upper $y_{k,t}^{(0)}$ bounds to exactly 0 or 1 can lead to errors, use, e.g., 1e-5 and 1-1e-5 instead.

▶ To use `nonParBayesSystemInferencePriorSets()` for a single component, you can set up a system with one component only:

```r
onecomp <- graph.formula(s -- 1 -- t)
onecomp <- setCompTypes(onecomp, list("A" = 1))
onecompss <- computeSystemSurvivalSignature(onecomp)
```

The component type name is arbitrary, but must be used in the `test.data` argument (as `test.data` is a named list).

▶ You can use `nonParBayesSystemInferencePriorSets()` to calculate the set of prior system reliability functions by setting

```r
test.data = list(A = NULL, B = NULL, C = NULL)
```
Exercise 1: Single Component Reliability Analysis

- Simulate 10 failure times from Gamma(5, 1).
- Calculate and graph the (sets of) prior and posterior reliability functions for...
  
  (a) a (discrete) precise prior, where the prior expected reliability values follow $1 - \text{plnorm}(t, 1.5, 0.25)$, and the prior strength is 5 for all $t$.
  
  (b) a vacuous set of priors, with prior strength interval $[1, 5]$. (Try out other prior strength intervals! Why is the lower bound irrelevant here?)
  
  (c) a set of priors based on an expert’s opinion that the functioning probability is, at time 2, between 0.9 and 0.8, and between 0.6 and 0.3 at time 5. Use $[1, 2]$ and $[10, 20]$ for the prior strength interval.
  
  (d) a set of priors, where $y_{k,t}^{(0)}$ follows $1-\text{pgamma}(t, 3, 4)$, $\bar{y}_{k,t}^{(0)}$ follows $1-\text{pgamma}(t, 3, 2)$, and the prior strength interval is $[1, 10]$. 
Exercise 2: System Reliability Analysis

- Define a system with multiple types of components using \texttt{graph.formula()} and \texttt{setCompTypes()}. You can take the ‘bridge’ system, the braking system, or invent your own system.

- For prior sets of component reliability functions, you can use the sets from the previous exercise, or think of your own.

- Simulate test data for the components such that they are, from the viewpoint of the component prior, ... 
  - as expected,
  - surprisingly early,
  - surprisingly late.

What is the effect on the posterior set of system reliability functions?

- Vary the sample size and the prior strength interval. What is the effect on the posterior set of system reliability functions?